Embedded Systems
2012/13

Basilica di Santa Maria di Collemaggio, 1287, L'Aquila

Lecture 6
Existence of Symbolic Models

Maria Domenica Di Benedetto
Giordano Pola
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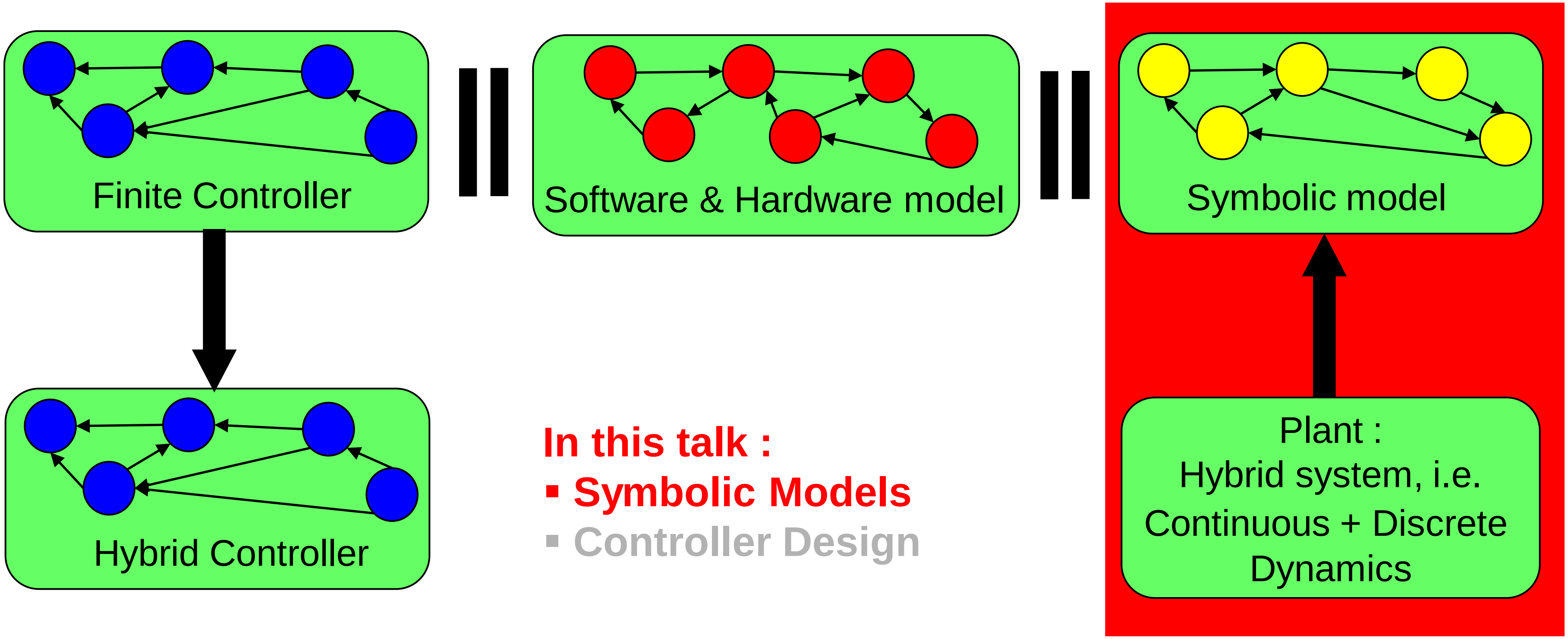
University of L'Aquila

Correct-by-Design Embedded Control Software

Synthesis through a three phase process:

1. Construct the finite model T of the plant system Σ
2. Synthesize a finite controller C solving the specification S on T
3. Synthesize a controller C' for Σ on the basis of C

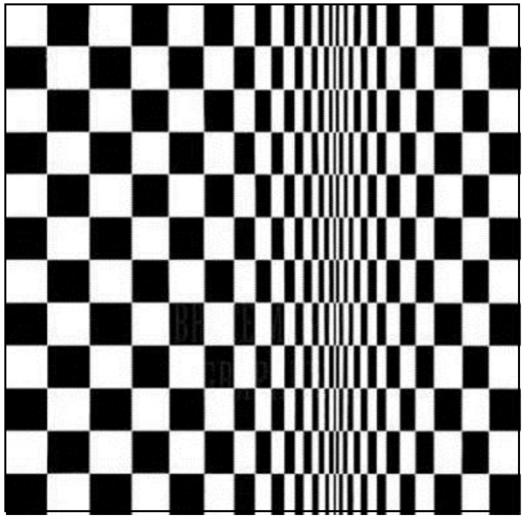


In this talk :

- **Symbolic Models**
- **Controller Design**

Continuous and Hybrid Systems

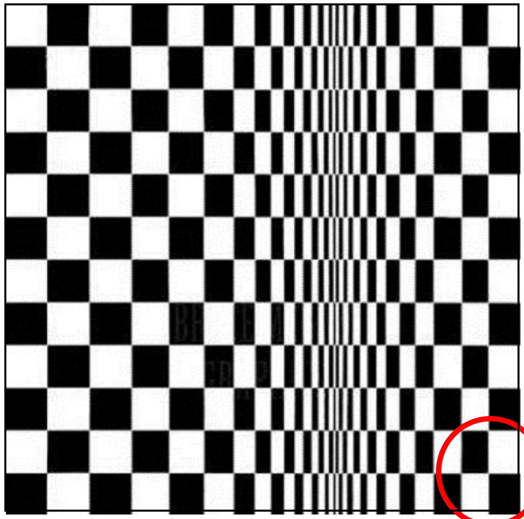
Salvador Dali, The Temptation of St. Anthony



Bridget Riley, Movement in Squares

Continuous and Hybrid Systems

Salvador Dali, The Temptation of St. Anthony



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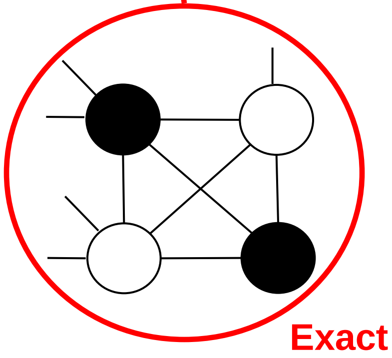

Continuous and Hybrid Systems

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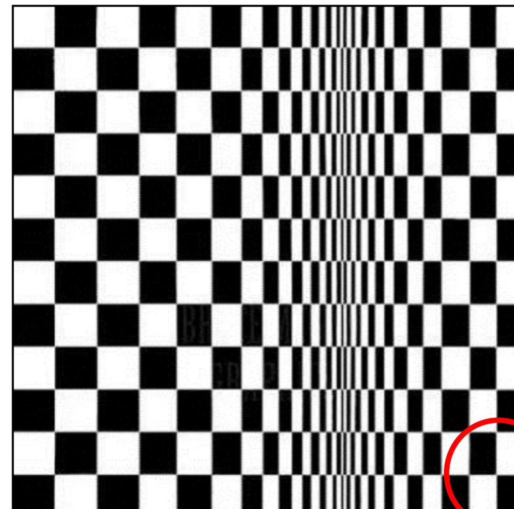
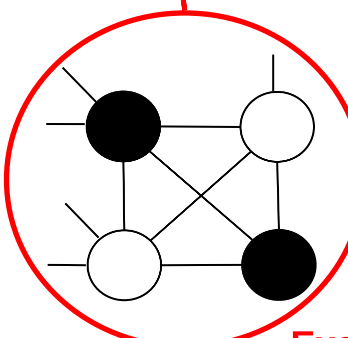


Bridget Riley, Movement in Squares

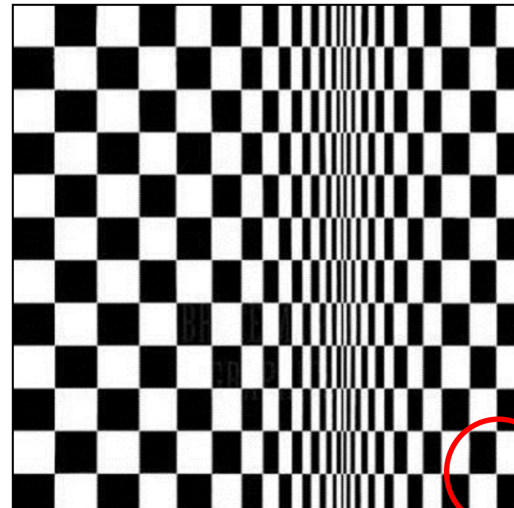
Exact ?



Exact

Continuous and Hybrid Systems

Salvador Dali, The Temptation of St. Anthony



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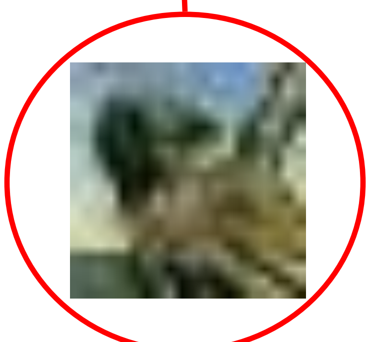
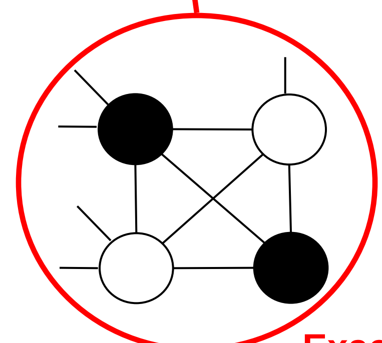


Approximated



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Continuous and Hybrid Systems

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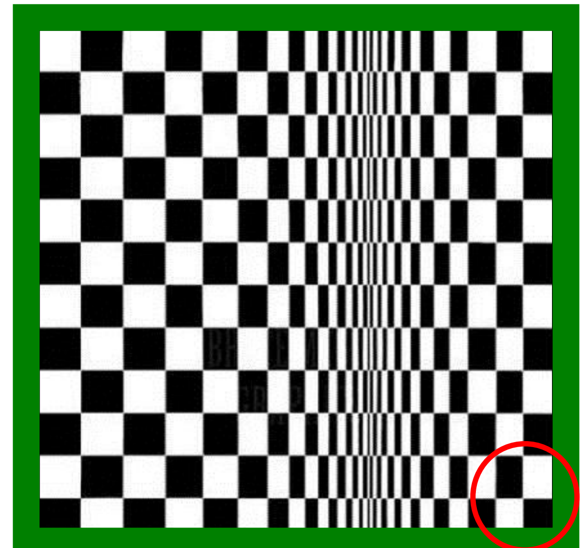

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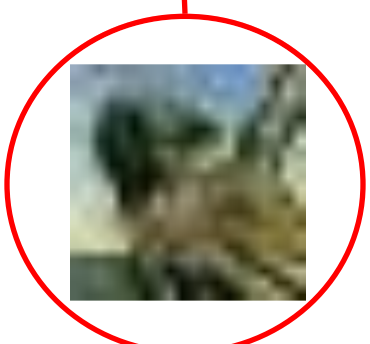
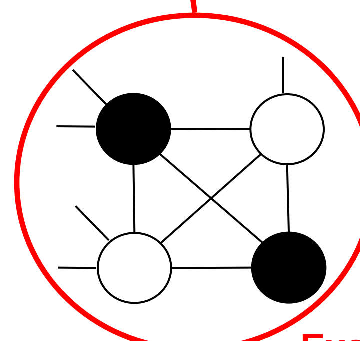


Approximated

There are systems that:

- Admit symbolic models
- Can be approximated by symbolic models
- Do not admit symbolic models



Exact

Continuous and Hybrid Systems

There are systems that:

- Admit symbolic models
  - Autonomous Systems (no input)
    - Timed Automata *[Alur & Dill, 1992]*
    - Multirate Automata *[Alur et al., 1993]*
    - Rectangular Automata *[Henzinger et al., 1998]*
    - o-minimal hybrid systems *[Lafferriere et al., 2000]*
  - Discrete-time controllable linear systems *[Tabuada & Pappas]*

- Can be approximated by symbolic models
  - Incrementally stable nonlinear control systems *[Pola et al., 2008]*
  - Incrementally stable switched nonlinear systems *[Girard et al., 2008]*
  - Incrementally stable nonlinear control systems with disturbances *[Pola & Tabuada 2009], [Borri et al. 2012]*
  - Incrementally stable nonlinear time-delay systems *[Pola et al. 2009]*
  - *Stable and Unstable Networked Control Systems [Borri et al. 2012]*

- Do not admit symbolic models

Continuous and Hybrid Systems

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- Do not admit symbolic models

Symbolic models for nonlinear control systems

Two key ingredients:

Incremental stability

A Lyapunov Approach to Incremental Stability Properties

David Angeli

***Abstract*—This paper deals with several notions of incremental stability. In other words, the focus is on stability of trajectories with respect to one another, rather than with respect to some attractor. The aim is to present a framework for understanding such questions fully compatible with the well-known input-to-state stability (ISS) approach. Applications of the newly introduced stability notions are also discussed.**

***Index Terms*—Lyapunov methods, observers, stability, synchronization.**

I. INTRODUCTION

INPUT-TO-STATE stability (ISS) has proven a valid instrument in order to study questions of robust stability for finite-dimensional nonlinear systems. One reason for that is the possibility of dealing, at the same time, with a body of theory which nicely extends to nonautonomous systems the classic Lyapunov while still allowing for input–output descriptions of the system behavior, [11], [21], [27], [29]. In this way tools such as small-gain theorems [12] and Lyapunov dissipation inequalities [26] have come together in a unified framework which bridges the gap between the state-space and input–output approaches.

Stability properties are described through the use of comparison functions, the so called class $\mathcal{K}$ and $\mathcal{KL}$ functions, which can be thought of as nonlinear versions of linear gains and exponentially fading transients. This approach naturally leads to stability notions which are invariant with respect to nonlinear changes of coordinates [9] and at the same time avoids the use of the $\varepsilon - \delta$ formalism which is usually less intuitive. A similar way of thinking was exploited in order to study detectability questions, [15], [28].

The quest for nonlinear analogs of the separation principle already involved input-to-state stability as one of the ingredients, [20], [32], [30]. As a matter of fact, it is especially looking at the issue of state-detection and observer synthesis that it becomes relevant to understand which systems may enjoy incremental stability properties. In other words our focus is on systems whose trajectories converge to one another, besides being attracted toward some equilibrium position. Works along these lines have recently appeared in the literature, [5], [23], together with some examples of applications, [8], [18]. As a matter of fact, the notion of incremental input-to-state stability that will be introduced, can be thought of also as an "open-loop observability" property, that is as the possibility of designing an observer for the system which only processes past input data. It is well-known that for linear systems such a property is equivalent to asymptotic stability. It is indeed a much stronger property when dealing with nonlinear ones.

As already pointed out, our aim is to present such notions in a framework compatible with the ISS approach. The paper is organized as follows: in Sections II and III we study robust incremental global asymptotic stability and prove a Lyapunov characterization of such a property; in Section IV we introduce incremental ISS and give some results of general interest, whereas a Lyapunov characterization of the property is provided in Section V; Section VI shows a couple of examples involving incremental ISS and conclusions are given in Section VII.

II. LYAPUNOV CHARACTERIZATIONS OF INCREMENTAL STABILITY

Let us consider dynamical systems of the following form:

$$\dot{x} = f(x, d) \tag{1}$$

with state $x \in \mathbb{R}^n$ and input d, here seen as a disturbance rather than a control, taking values on a closed set $\mathcal{D} \subset \mathbb{R}^m$. By input signal we mean any measurable, locally essentially bounded function of time and we denote the set of such functions by $\mathcal{M}_{\mathcal{D}}$.

We are interested in characterizing in terms of Lyapunov dissipation inequalities the following property of solutions of (1).

Definition 2.1: We say that (1) is incremental globally asymptotically stable (δGAS) if there exists a function β of class $\mathcal{KL}$ so that for all $d \in \mathcal{M}_{\mathcal{D}}$, all $\xi, \eta \in \mathbb{R}^n$ and all $t \geq 0$ the following holds

$$|x(t, \xi, d) - x(t, \eta, d)| \leq \beta(|\xi - \eta|, t). \tag{2}$$

□

It is convenient to recast the notion of incremental stability as a standard problem of uniform global asymptotic stability with respect to sets. For the sake of completeness we recall the definition of uniform GAS.

Definition 2.2: We say that (1) is globally asymptotically stable (GAS) with respect to a closed set $\mathcal{A}$ if there exists a function β of class $\mathcal{KL}$ so that for all $d \in \mathcal{M}_{\mathcal{D}}$, all $\xi \in \mathbb{R}^n$ and all $t \geq 0$ the following holds:

$$|x(t, \xi, d)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t). \tag{3}$$

□

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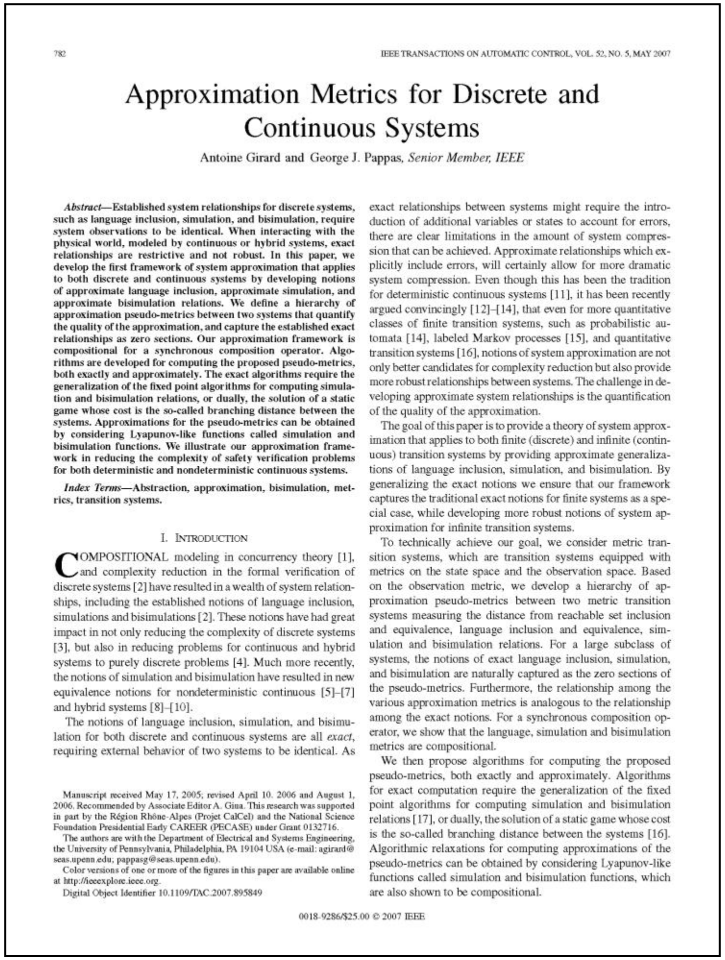
Approximate bisimulation



Approximation Metrics for Discrete and Continuous Systems

Antoine Girard and George J. Pappas, *Senior Member, IEEE*

***Abstract*—Established system relationships for discrete systems, such as language inclusion, simulation, and bisimulation, require system observations to be identical. When interacting with the physical world, modeled by continuous or hybrid systems, exact relationships are restrictive and not robust. In this paper, we develop the first framework of system approximation that applies to both discrete and continuous systems by developing notions of approximate language inclusion, approximate simulation, and approximate bisimulation relations. We define a hierarchy of approximation pseudo-metrics between two systems that quantify the quality of the approximation, and capture the established exact relationships as zero sections. Our approximation framework is compositional for a synchronous composition operator. Algorithms are developed for computing the proposed pseudo-metrics, both exactly and approximately. The exact algorithms require the generalization of the fixed point algorithms for computing simulation and bisimulation relations, or dually, the solution of a static game whose cost is the so-called branching distance between the systems. Approximations for the pseudo-metrics can be obtained by considering Lyapunov-like functions called simulation and bisimulation functions. We illustrate our approximation framework in reducing the complexity of safety verification problems for both deterministic and nondeterministic continuous systems.**

***Index Terms*—Abstraction, approximation, bisimulation, metrics, transition systems.**

I. INTRODUCTION

COMPOSITIONAL modeling in concurrency theory [1], and complexity reduction in the formal verification of discrete systems [2] have resulted in a wealth of system relationships, including the established notions of language inclusion, simulations and bisimulations [2]. These notions have had great impact in not only reducing the complexity of discrete systems [3], but also in reducing problems for continuous and hybrid systems to purely discrete problems [4]. Much more recently, the notions of simulation and bisimulation have resulted in new equivalence notions for nondeterministic continuous [5]–[7] and hybrid systems [8]–[10].

The notions of language inclusion, simulation, and bisimulation for both discrete and continuous systems are all *exact*, requiring external behavior of two systems to be identical. As exact relationships between systems might require the introduction of additional variables or states to account for errors, there are clear limitations in the amount of system compression that can be achieved. Approximate relationships which explicitly include errors, will certainly allow for more dramatic system compression. Even though this has been the tradition for deterministic continuous systems [11], it has been recently argued convincingly [12]–[14], that even for more quantitative classes of finite transition systems, such as probabilistic automata [14], labeled Markov processes [15], and quantitative transition systems [16], notions of system approximation are not only better candidates for complexity reduction but also provide more robust relationships between systems. The challenge in developing approximate system relationships is the quantification of the quality of the approximation.

The goal of this paper is to provide a theory of system approximation that applies to both finite (discrete) and infinite (continuous) transition systems by providing approximate generalizations of language inclusion, simulation, and bisimulation. By generalizing the exact notions we ensure that our framework captures the traditional exact notions for finite systems as a special case, while developing more robust notions of system approximation for infinite transition systems.

To technically achieve our goal, we consider metric transition systems, which are transition systems equipped with metrics on the state space and the observation space. Based on the observation metric, we develop a hierarchy of approximation pseudo-metrics between two metric transition systems measuring the distance from reachable set inclusion and equivalence, language inclusion and equivalence, simulation and bisimulation relations. For a large subclass of systems, the notions of exact language inclusion, simulation, and bisimulation are naturally captured as the zero sections of the pseudo-metrics. Furthermore, the relationship among the various approximation metrics is analogous to the relationship among the exact notions. For a synchronous composition operator, we show that the language, simulation and bisimulation metrics are compositional.

We then propose algorithms for computing the proposed pseudo-metrics, both exactly and approximately. Algorithms for exact computation require the generalization of the fixed point algorithms for computing simulation and bisimulation relations [17], or dually, the solution of a static game whose cost is the so-called branching distance between the systems [16]. Algorithmic relaxations for computing approximations of the pseudo-metrics can be obtained by considering Lyapunov-like functions called simulation and bisimulation functions, which are also shown to be compositional.

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Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

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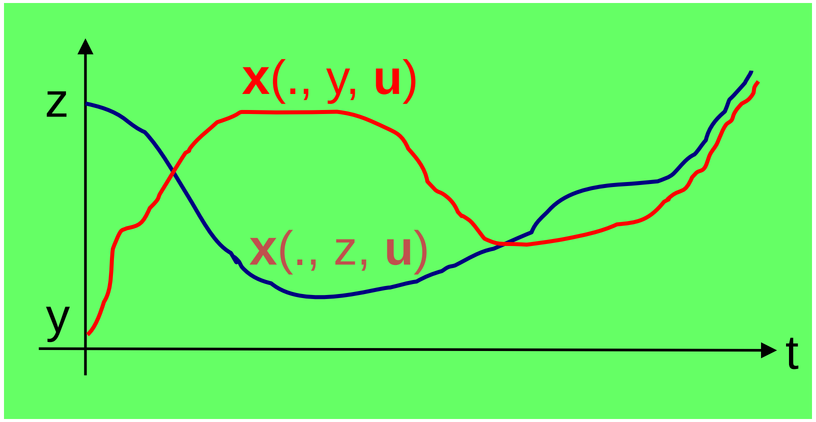
Incremental Global Asymptotic Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Globally Asymptotically Stable (δ-GAS) if there exists a KL function β so that for any $t \geq 0$, $y,z \in R^n$ and $\mathbf{u}$

$$\| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{u}) \| \leq \beta(\|y-z\|, t)$$



Additional details from D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

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$$|| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{u}) || \leq \boldsymbol{\beta}(||y-z||, t)$$

Theorem:

A control system $\dot{x} = f(x,u)$ is δ-GAS if there exists a smooth function $V : R^n \times R^n \rightarrow R^+$ and K_∞ functions α_1, α_2, ρ such that:

i) $\alpha_1 (|| x - y ||) \leq V(x, y) \leq \alpha_2 (|| x – y ||)$ for all $x,y \in R^n$

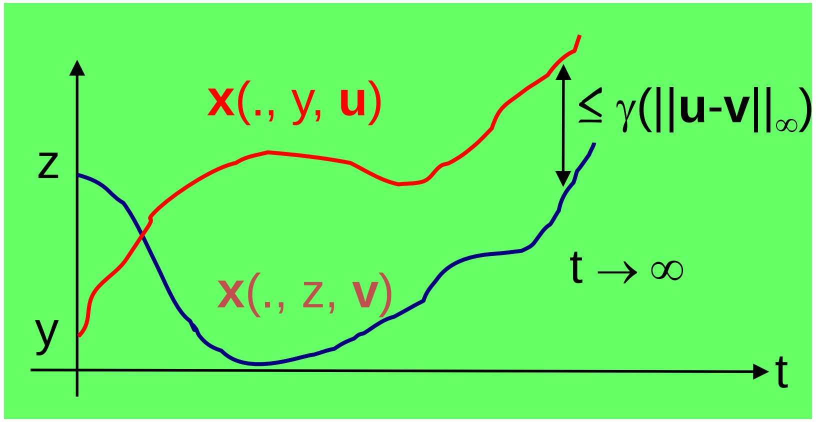
ii) $dV/dx\, f(x,u) + dV/dy\, f(y,u) < - \rho (|| x – y ||)$ for all $x,y \in R^n$ and $u \in R^m$

Additional details in D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Incremental Input-State Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Input-to-State Stable (δ-ISS) if there exist a KL function β and a K_∞ function γ so that for any $t \geq 0$, $y, z \in R^n$ and $\mathbf{u}$, $\mathbf{v}$

$$\| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{v}) \| \leq \beta(\|y - z\|, t) + \gamma(\|\mathbf{u} - \mathbf{v}\|_\infty)$$



Further details in D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Incremental Input-State Stability

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$$|| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{v}) || \leq \boldsymbol{\beta}(||y - z||, t) + \gamma(||\mathbf{u} - \mathbf{v}||_\infty)$$

Theorem:

A control system $\dot{x} = f(x,u)$ is δ-ISS if there exists a smooth function $V : R^n \times R^n \to R^+$ and K_∞ functions α_1, α_2, ρ, σ such that:

i) $\alpha_1 (||x - y||) \leq V(x, y) \leq \alpha_2 (||x - y||)$ for all $x,y \in R^n$

ii) $dV/dx\, f(x,u) + dV/dy\, f(y,v) < - \rho (||x - y||) + \sigma (||u - v||)$ for all $x,y \in R^n$ and $u,v \in R^m$

Further details from D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Connections among Stability Notions

$$
\begin{tikzcd}
\delta\text{-ISS} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "f(0,0)=0"'] & \delta\text{-GAS} \arrow[d, Rightarrow, "f(0,0)=0"] \\
\text{ISS} \arrow[r, Rightarrow] & \text{GAS}
\end{tikzcd}
$$

Homework:

1) Prove such connections!
2) How do these notions specialize to the case of linear control systems?

Bisimulation equivalence

Given $T_1 = (Q_1, L_1, \longrightarrow_1, O_1, H_1)$ and $T_2 = (Q_2, L_2, \longrightarrow_2, O_2, H_2)$ with $O_1 = O_2$, a relation

$$R \subseteq Q_1 \times Q_2$$

is a ***bisimulation relation*** between T_1 and T_2 if for all $(q_1, q_2) \in R$

- $H_1(q_1) = H_2(q_2)$
- $q_1 \xrightarrow{l_1}_1 p_1$ in T_1 implies existence of $q_2 \xrightarrow{l_2}_2 p_2$ in T_2 so that $(p_1, p_2) \in R$
- $q_2 \xrightarrow{l_2}_2 p_2$ in T_2 implies existence of $q_1 \xrightarrow{l_1}_1 p_1$ in T_1 so that $(p_1, p_2) \in R$

LTSs T_1 and T_2 are ***bisimilar*** if $\pi|_{Q_1}(R) = Q_1$ and $\pi|_{Q_2}(R) = Q_2$

Approximate bisimulation equivalence

Given $T_1 = (Q_1, L_1, \longrightarrow_1, O_1, H_1)$ and $T_2 = (Q_2, L_2, \longrightarrow_2, O_2, H_2)$ with $O_1 = O_2$, and a precision $\varepsilon > 0$, a relation

$$R \subseteq Q_1 \times Q_2$$

is an ***approximate bisimulation relation*** between T_1 and T_2 if for all $(q_1, q_2) \in R$

- $\cancel{H_1(q_1) = H_2(q_2)}$ $d(H_1(q_1), H_2(q_2)) \leq \varepsilon$
- $q_1 \xrightarrow{l_1}_1 p_1$ in T_1 implies existence of $q_2 \xrightarrow{l_2}_2 p_2$ in T_2 so that $(p_1, p_2) \in R$
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We now consider a modified version of bisimulation where outputs need not to coincide but to be close, up to a given precision ε

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We now consider a modified version of bisimulation where outputs need not to coincide but to be close, up to a given precision ε

… is approximate bisimulation equivalence an equivalence relation?

Approximate bisimulation equivalence

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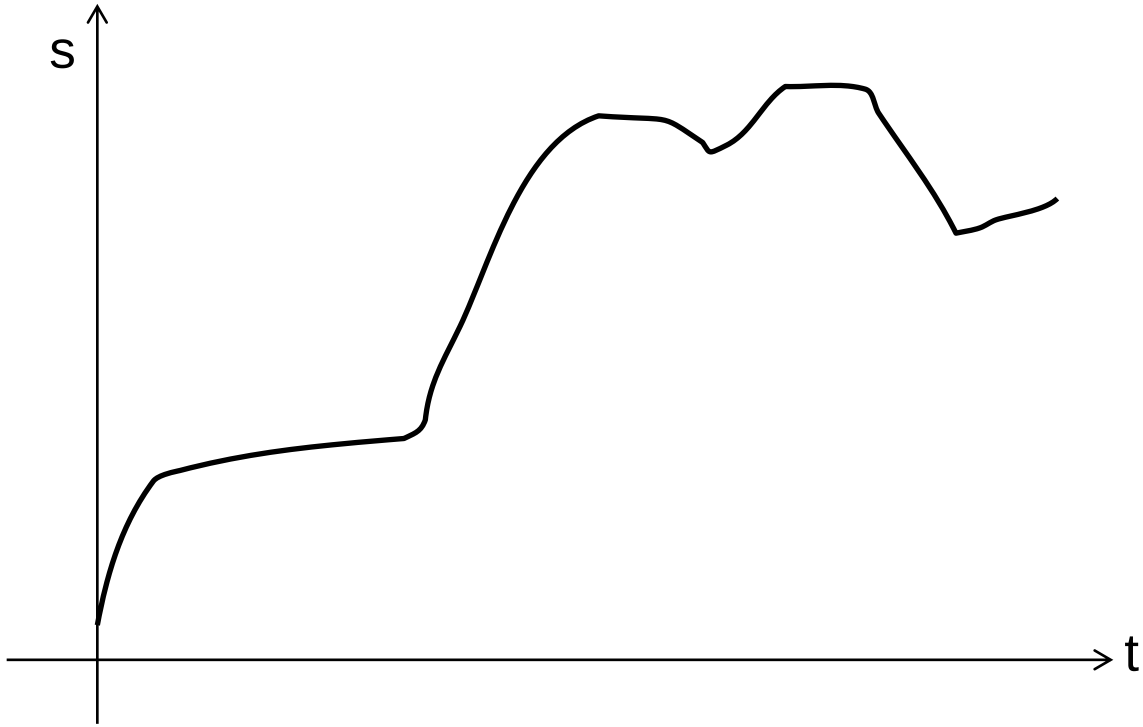
We now consider a modified version of bisimulation where outputs need not to coincide but to be close, up to a given precision ε

What about approximate simulation?

Existence of symbolic models

"Discretization" of a signal s: R $\rightarrow$ R



Existence of symbolic models

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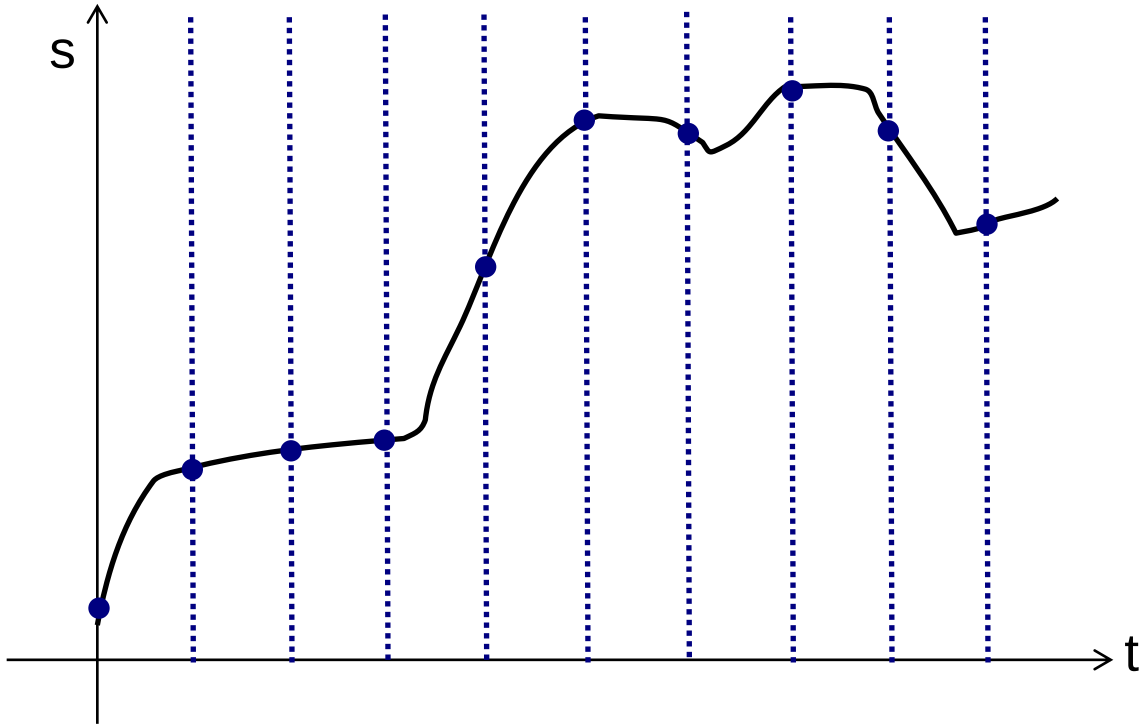


1. **Time quantization**

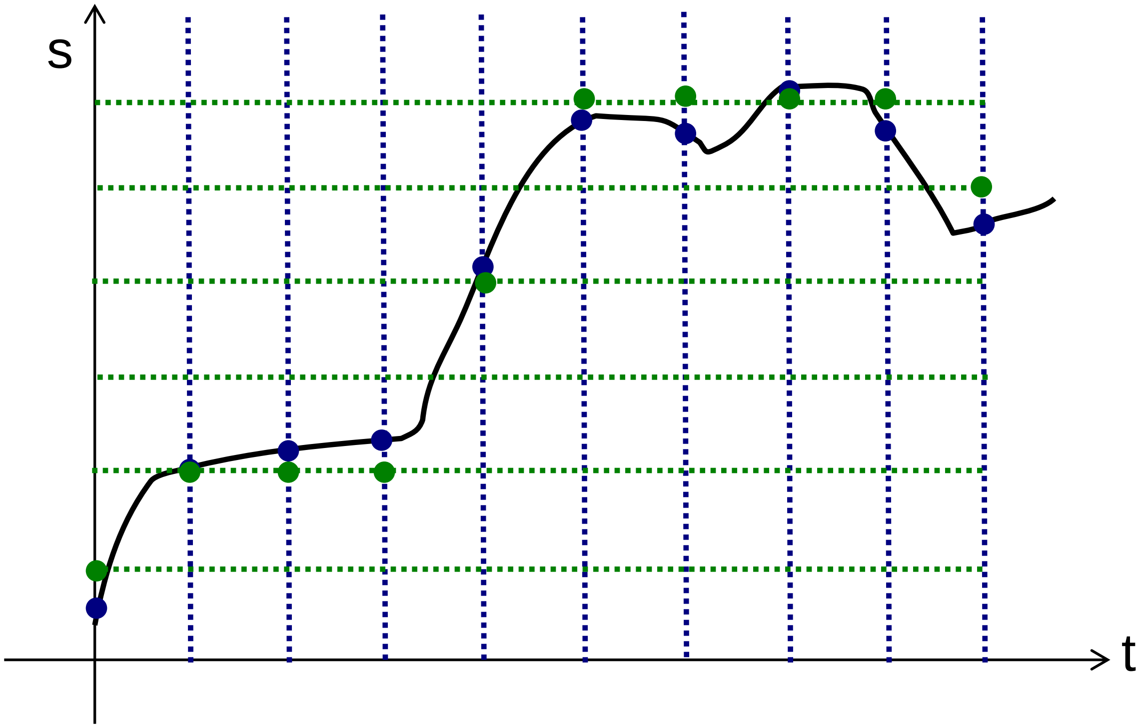
   from s: R $\rightarrow$ R

   to r: N $\rightarrow$ R

   Signal r is not "symbolic" !

Existence of symbolic models

"Discretization" of a signal s: R → R



1. **Time quantization**

   from s: R → R

   to r: N → R

   Signal r is not "symbolic" !

2. **Space quantization**

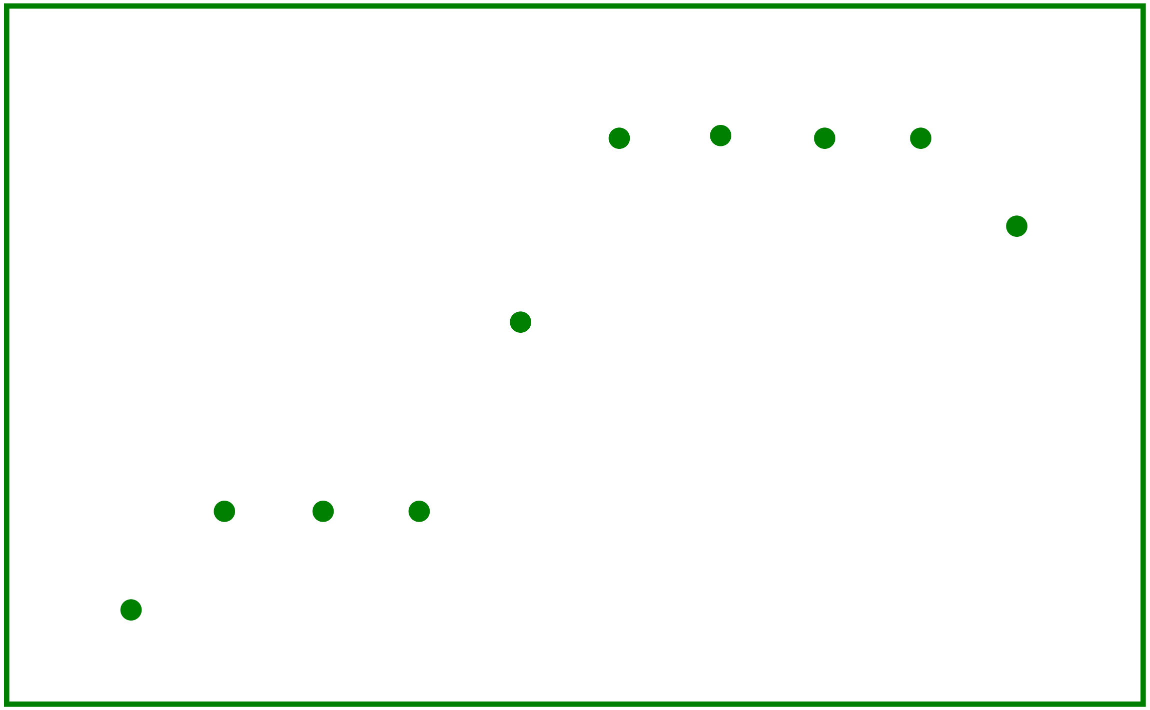
   from r: N → R

   to q: N → $\{q_1, q_2, \ldots, q_n\}$

   Signal q is "symbolic" !

Existence of symbolic models

“Discretization” of a signal s: $R \rightarrow R$



1. **Time quantization**

   from s: $R \rightarrow R$

   to r: $N \rightarrow R$

   Signal r is not “symbolic” !

2. **Space quantization**

   from r: $N \rightarrow R$

   to q: $N \rightarrow \{q_1, q_2, \ldots, q_n\}$

   Signal q is “symbolic” !

So that I can approximate signal s by 10 symbols!

From symbolic models for signals to symbolic models for nonlinear control systems…

Existence of symbolic models

Given a nonlinear control system Σ

$$\dot{x} = f(x,u) \quad , \quad x \in R^n, u \in R^m$$

and a time $\tau > 0$ consider:

$$T_\tau(\Sigma) = (Q, L, \longrightarrow, O, H)$$

where:

- $Q = R^n$
- L is the collection of control signals $u: [0,\tau] \rightarrow R^m$
- $p \xrightarrow{u} q$, if $x(\tau,p,u)=q$
- $O = R^n$
- H is the identity function

$T_\tau(\Sigma)$ can be thought of as a time discretization of $T(\Sigma)$ or equivalently of Σ

$T_\tau(\Sigma)$ is not a symbolic model because Q and L are infinite sets!

Existence of symbolic models

Theorem Consider a nonlinear control system Σ

$$\dot{x} = f(x,u)\ , \quad x \in R^n,\ u \in R^m$$

If Σ is δ-GAS then for any precision $\varepsilon > 0$ there exists a time $\tau > 0$ and a **countable** transition system T so that $T_\tau(\Sigma)$ and T are approximately bisimilar with precision ε.

δ-GAS $\Rightarrow$ **existence of symbolic models**

Existence of symbolic models

Theorem Consider a nonlinear control system Σ

$$\dot{x} = f(x,u) \;, \quad x \in R^n,\ u \in R^m$$

If Σ is δ-GAS then for any precision $\varepsilon > 0$ there exists a time $\tau > 0$ and a **countable** transition system T so that $T_\tau(\Sigma)$ and T are approximately bisimilar with precision ε.

Corollary Consider a nonlinear control system Σ

$$\dot{x} = f(x,u) \;, \quad x \in X,\ u \in U$$

If Σ is δ-GAS and X is bounded, then for any precision $\varepsilon > 0$ there exists a time $\tau > 0$ and a **finite/symbolic** transition system T so that $T_\tau(\Sigma)$ and T are approximately bisimilar with precision ε.

Proof (Sketch)

Which quantization parameters
do I need to construct a symbolic model?

Proof (Sketch)

Consider the following parameters:

- $\tau > 0$ sampling time
- $\eta > 0$ state space quantization
- $\mu > 0$ input space quantization

and define $T_{\tau,\eta,\mu}(\Sigma) = (Q, L, \longrightarrow, O, H)$

Proof (Sketch)

Consider the following parameters:

- $\tau > 0$ sampling time
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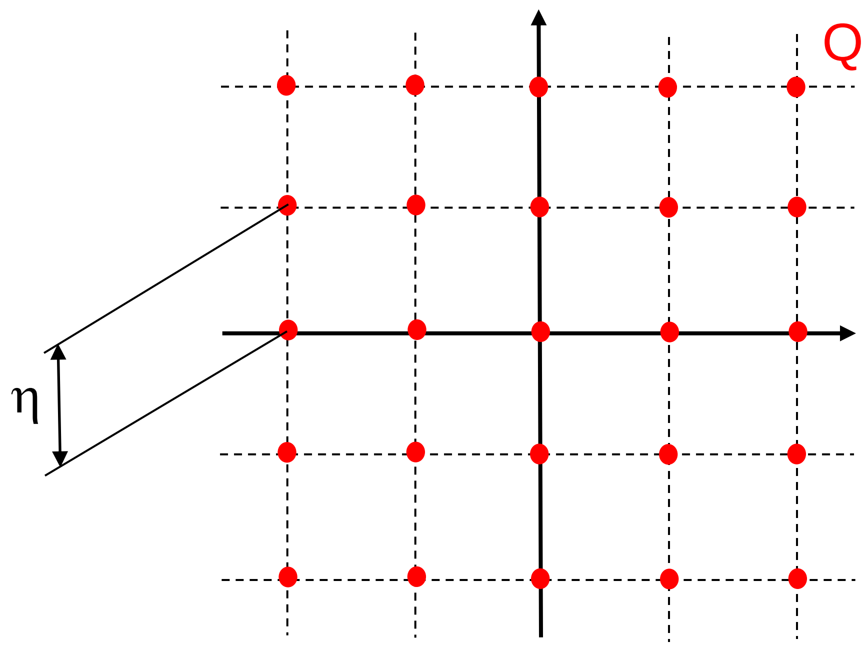
and define $T_{\tau,\eta,\mu}(\Sigma) = (Q, L, \longrightarrow, O, H)$

where:

- $Q = [R^n]_\eta = \eta Z^n$



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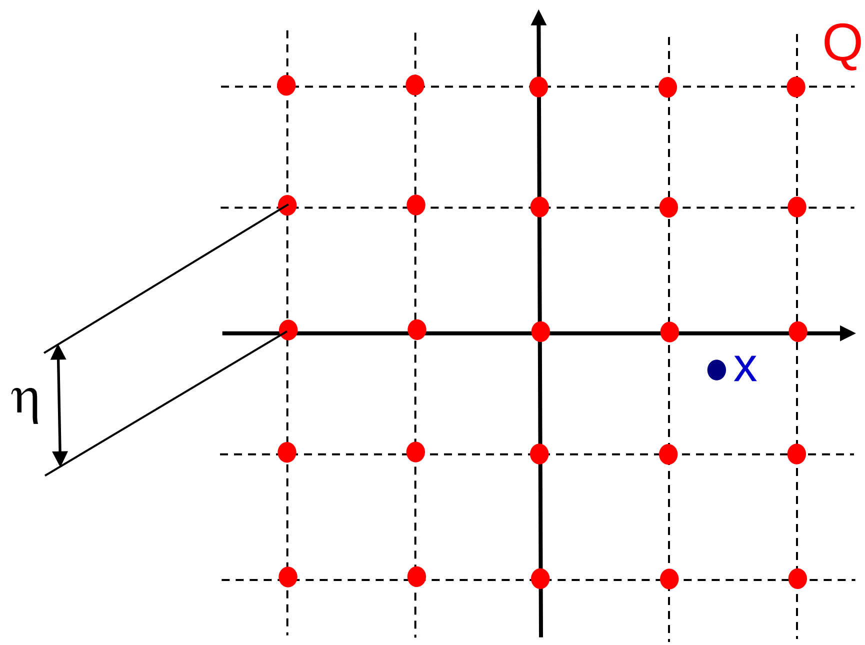
- $Q = [R^n]_\eta = \eta Z^n$

$\boxed{\begin{array}{l} Q \text{ approximates } R^n \text{ with precision } \eta \\ \forall x \in R^n \end{array}}$



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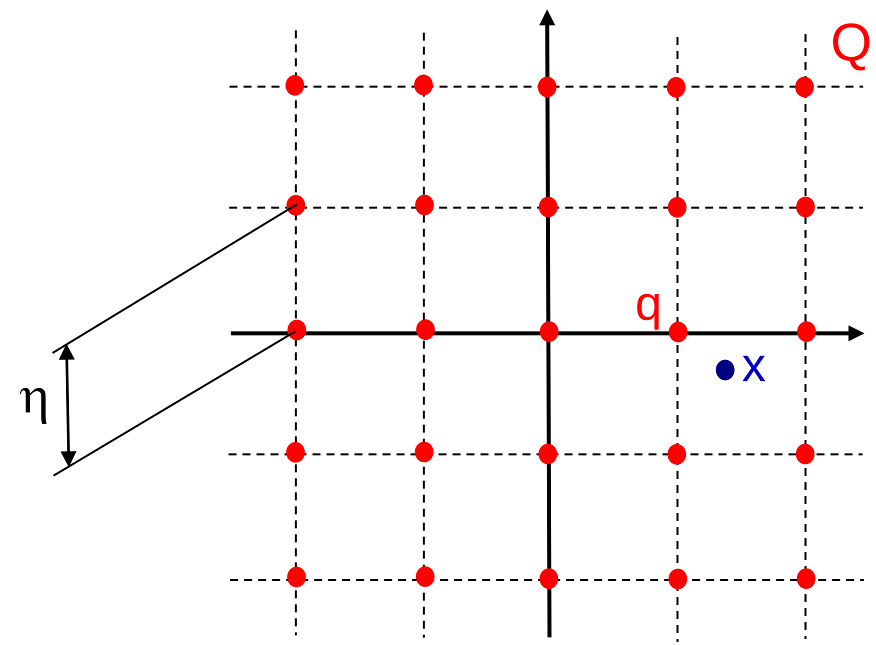
- $Q = [R^n]_\eta = \eta Z^n$

Q approximates R^n with precision η

$\forall x \in R^n \; \exists q \in Q$



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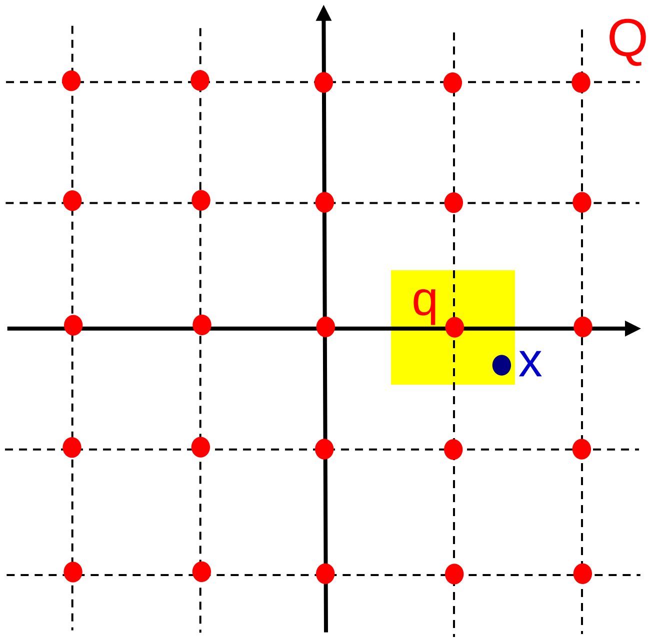

Q approximates R^n with precision η

$\forall x \in R^n \; \exists q \in Q$ s.t. $|x - q|_\infty \leq \eta/2$

Proof (Sketch)

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- $\mu > 0$ input space quantization

and define $T_{\tau,\eta,\mu}(\Sigma) = (Q, L, \longrightarrow, O, H)$

where:

- $Q = [R^n]_\eta = \eta Z^n$
- $L = ?$

Proof (Sketch)

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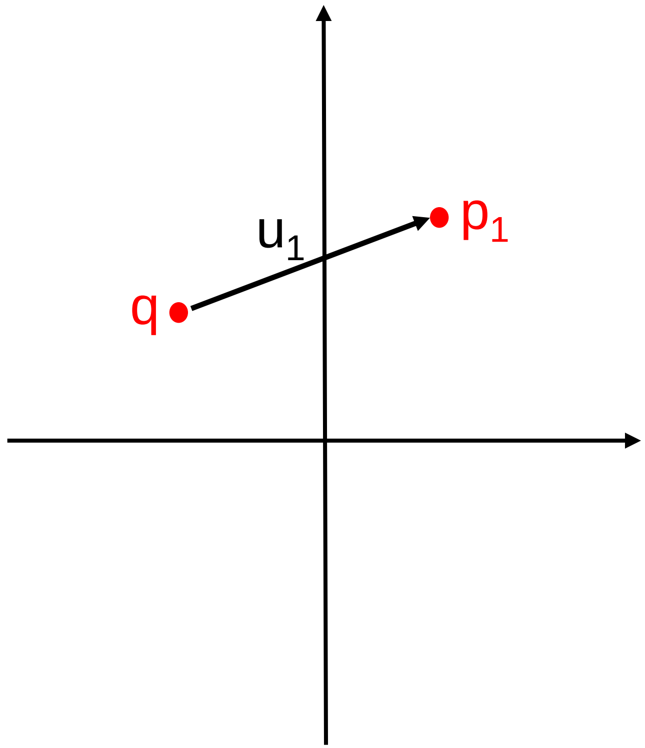
where:

- $Q = [R^n]_\eta = \eta Z^n$
- $L = \cup_{q \in Q} L(q)$ where:



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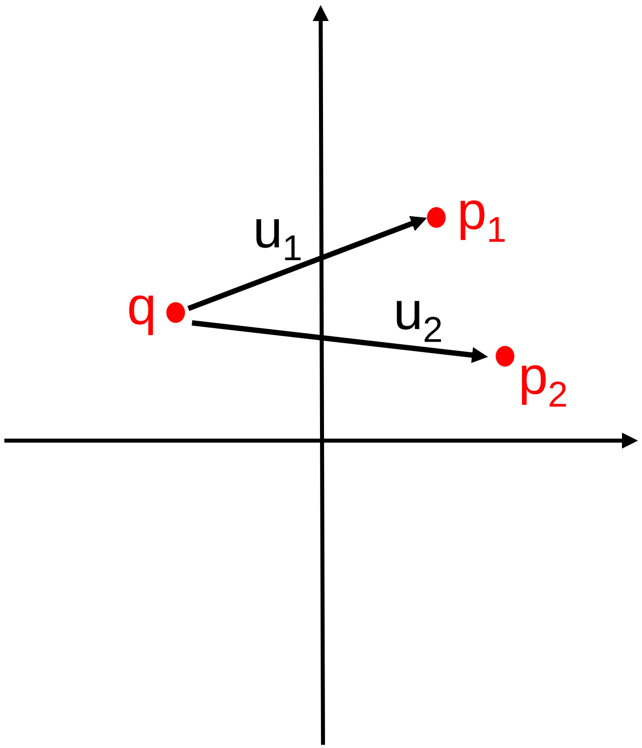

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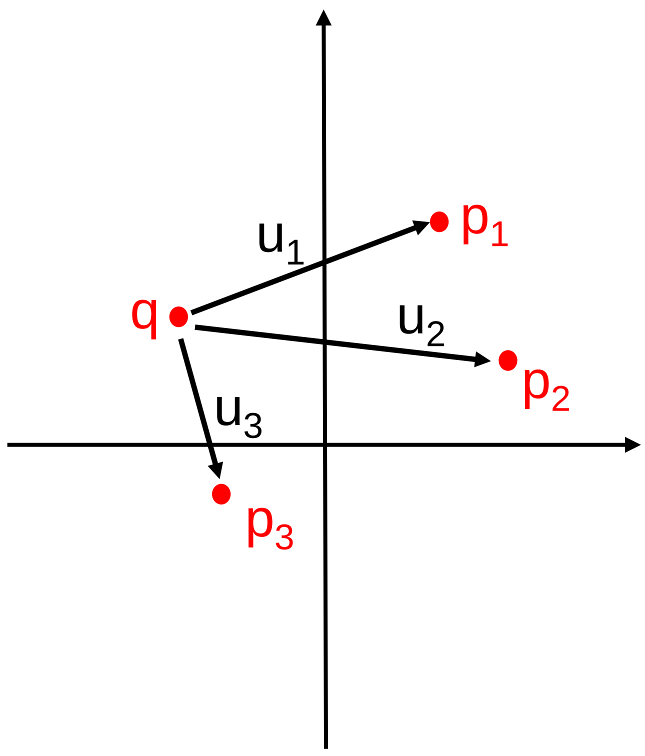

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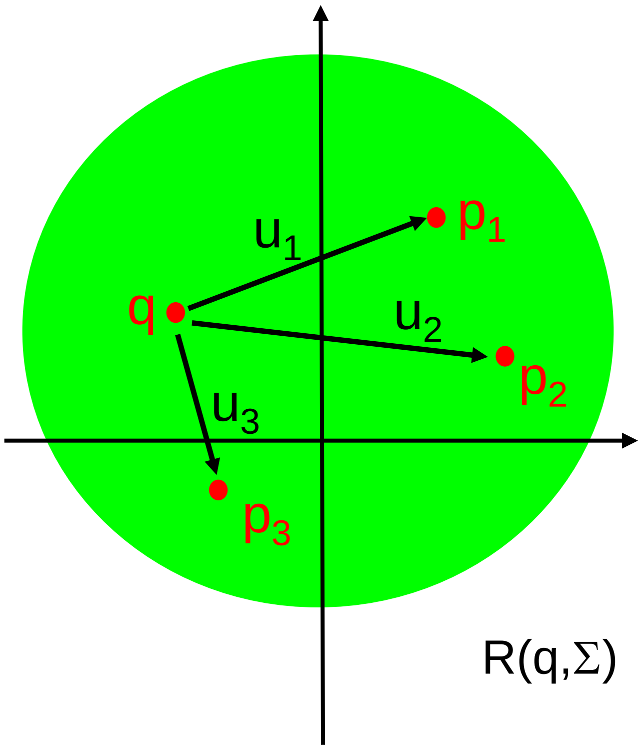

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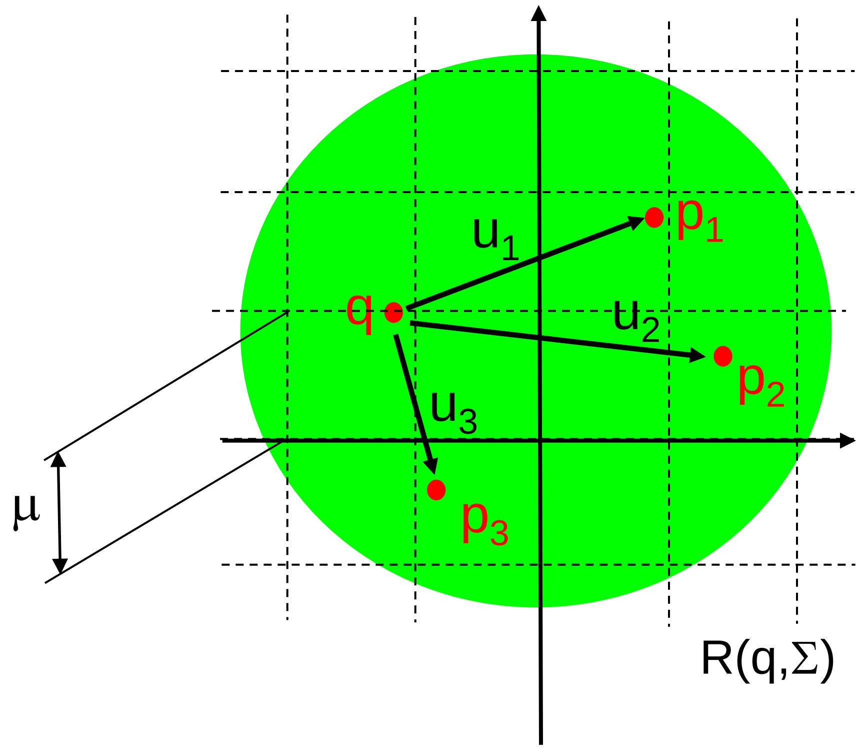

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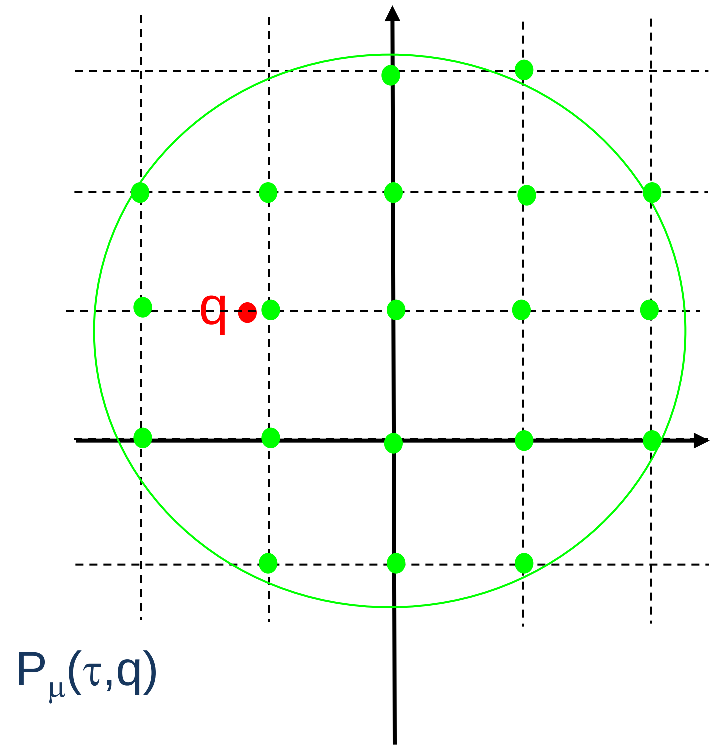

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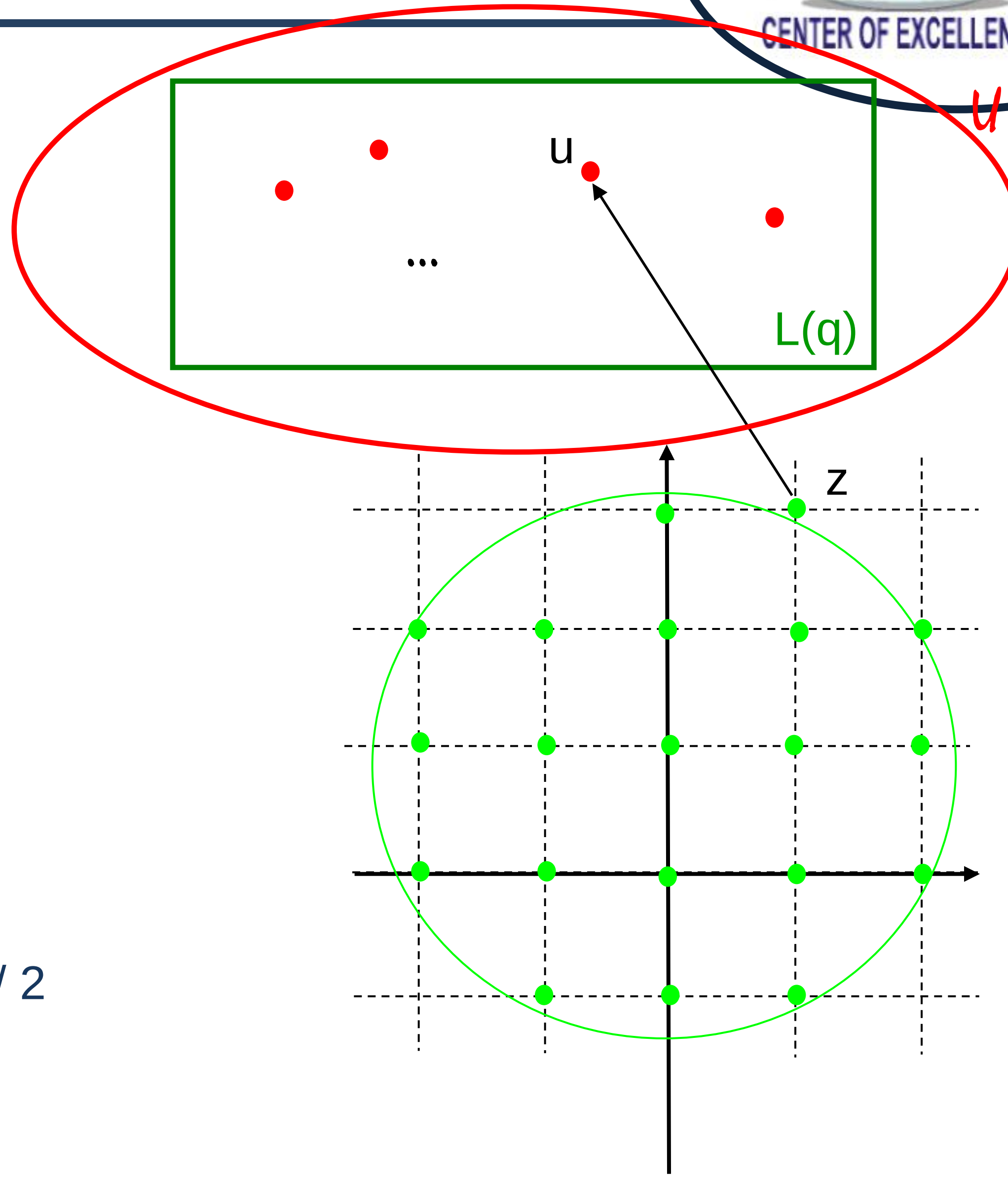

$$\| z - x(\tau, q, u) \| \leq \mu / 2$$

Proof (Sketch)

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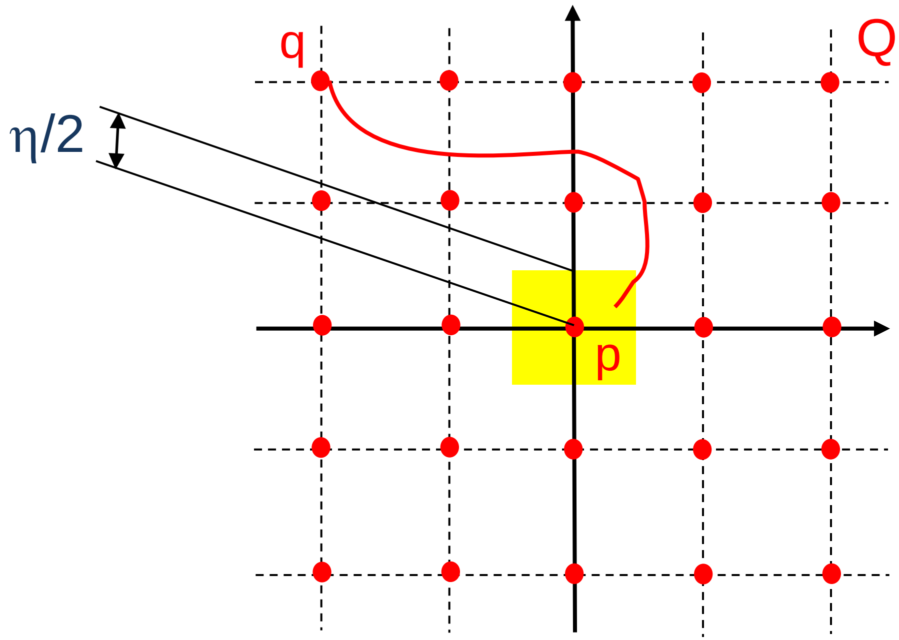
where:

- $Q = [R^n]_\eta = \eta Z^n$
- $L = \cup_{q \in Q} L(q)$ where:
- $q \xrightarrow{u} p$, if $| x(\tau,q,u) - p |_\infty \leq \eta/2$
- $O = R^n$
- H is the identity function



Labelled transition system $T_{\tau,\eta,\mu}(\Sigma)$ is countable